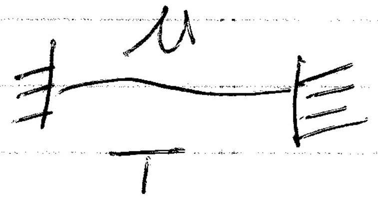
ii) <u>Non-trivial</u>

Derive NL equation for string with tension T (1D)



- end point fixed
$\delta y(0) = \delta y(L) = 0$
all t
- $\delta y(x, t_{2,1}) = 0$

$$\sum_i = \int dx$$

so
$$S = \int_{t_1}^{t_2} dt \int_0^L dx \, \mathcal{L}$$
Lagrangian density

G.C. $y(x,t)$

$$\mathcal{L} = \tfrac{1}{2}\mu\,(\partial_t y(x,t))^2 - \mathcal{U}$$

Now,
$$U = \int_0^L ds\, T - \int_0^L dx\, T \quad \text{(ref.)} \Rightarrow \text{cancel.}$$
potential energy stored in string

but $ds^2 = dx^2 + dy^2$
$= dx^2\left(1 + (dy/dx)^2\right)$

so $U = \int_0^L dx\, T\,(1+(dy/dx)^2)^{1/2}$

$\Rightarrow$

$$\mathcal{L} = 1/2\, \mu\, (\partial_t y(x,t))^2 - T\,(1+(dy/dx)^2)^{1/2}$$

so

$$S = \int dt \int_0^L dx \left\{ 1/2\, \mu\, (\partial_t y)^2 - T\,(1+(\partial_x y)^2)^{1/2} \right\}$$

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L \left\{ \frac{\partial \mathcal{L}}{\partial y_t}\, \delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\, \delta y_x \right\}$$

$$= \int_0^L dx\, \overbrace{\cancel{\frac{\partial \mathcal{L}}{\partial y_t}\, \delta y \Big|_{t_1}^{t_2}}}^{\text{no variation so}} - \int_{t_1}^{t_2} dt \int_0^L dx\, \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right)$$

$$+ \underbrace{\cancel{\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial y_x}\, \delta y \Big|_0^L dt}}_{\text{fixed e.p.}} - \int_{t_1}^{t_2} dt \int_0^L dx\, \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0$$

so

$$\delta S = 0 \qquad (PLA)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0}$$

Lagrange Eqn.

Now $\dfrac{\partial \mathcal{L}}{\partial y_t} = \mu y_t$

$$\frac{\partial \mathcal{L}}{\partial y_x} = \frac{-T \,\partial y/\partial x}{\left[1+\left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2}}$$

so

$$\boxed{\partial_t\left(\mu\, y_t\right) - \partial_x\left(\frac{T\,\partial y/\partial x}{\left[1+(\partial y/\partial x)^2\right]^{1/2}}\right) = 0}$$

- nonlinear eqn. for waves on clamped string.

- linearizing for μ, T const

$$\Rightarrow \boxed{\mu \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}}$$

wave eqn, as expected!

→ Conservation and Symmetry {Read Chapt. 2 L&L}

- Energy

→ will show if $\partial_t L = 0$ (no explicit time dependence)
i.e. homogeneity of time, unless broken by external input

⇒ energy conservation
homogeneity of time, for closed system.

Now $$\frac{dL}{dt} = \cancel{\frac{\partial L}{\partial t}} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$
[can shift $t=0$, arbitrarily]
$$= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$\partial L/\partial q = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \Rightarrow$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

so

$$\frac{d}{dt}\left(\dot{q} \frac{\partial L}{\partial \dot{q}} - L \right) = 0$$

so

$$E = \dot{q} \frac{\partial L}{\partial \dot{q}} - L \equiv \underline{\text{energy}}$$

- const. energy
- defines <u>conservative</u> system

i.e. one with $\partial_t L = 0$.

Note:- Hamiltonian H defined s/t

$$H = p\dot{q} - L$$

where $p = \partial L / \partial \dot{q}$ and eliminate $\dot{q}$ in favor p.

- in general $H \neq E$ ($H = E$ for conservative)
- can define H when $\partial_t L \neq 0$, so energy not conserved.

- Linear Momentum

→ for closed system, homogeneity of space ⇒ mechanical properties unchanged by any parallel displacement of the system in space

i.e. can shift origin of coordinate system, physics invariant upon $\underline{r} \to \underline{r} + \underline{\epsilon}$

so

$$\delta L = \sum_i \frac{\partial L}{\partial \underline{r}_i} \cdot \underline{\epsilon} = \underline{\epsilon} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i}$$

$$\delta L = 0 \Rightarrow \sum_i \frac{\partial L}{\partial \underline{r}_i} = 0 \text{ , so}$$

using Lagrange's Eqn.

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \underline{v}_i}\right) = 0$$

but recall: $\dfrac{\partial L}{\partial \underline{v}_i} = \underline{p}_i$ ⇒ momentum so $\frac{d}{dt}\underline{P} = 0$ $\qquad \underline{P} = \sum_i \partial L / \partial \underline{v}_i$

momentum conserved!

$\therefore$ for $\underline{F}_i = -\frac{\partial U}{\partial \underline{r}_i} = \frac{\partial L}{\partial \underline{r}_i} = 0$

gen. force, $= 0 \Leftrightarrow$ homog. of space.

gen. momentum conserved.

- <u>Angular</u> <u>Momentum</u>

$\Rightarrow$ <u>isotropy</u> of space, for closed system,

$\Rightarrow$ physics invariant under infinitesimal rotation.

<u>i.e.</u> $\delta \underline{\phi} \equiv$ vector infinitesimal rotation

$|\delta\phi| \rightarrow$ magnitude, $\frac{\delta \underline{\phi}}{|\delta\phi|} \rightarrow$ direction of axis

then upon rotation $\Rightarrow$

$\delta \underline{r} = \delta \underline{\phi} \times \underline{r}$

$\delta \underline{v} = \delta \underline{\phi} \times \underline{v}$ $\qquad$ velocities also change.

⇒ so, isotropy of space ⇒ $\delta L = 0$ upon infinitesimal rotation

so

$$\delta L\Big|_{\text{rotation}} = \sum_i \Big(\underbrace{\frac{\partial L}{\partial \underline{r}_i}}_{\dot{\underline{p}}_i} \cdot \delta \underline{r}_i + \underbrace{\frac{\partial L}{\partial \underline{v}_i}}_{\underline{p}_i} \cdot \delta \underline{v}_i \Big)$$

so symmetry under rotation ⇒

$$\delta L = 0 = \sum_i \left(\dot{\underline{p}}_i \cdot \delta \underline{\phi} \times \underline{r}_i + \underline{p}_i \cdot \delta \underline{\phi} \times \underline{v}_i \right)$$

re-arranging the terms:

$$\delta L = 0 = \delta \underline{\phi} \cdot \sum_i \Big(\underline{r}_i \times \dot{\underline{p}}_i + \overbrace{\dot{\underline{r}}_i}^{\underline{v}_i} \times \underline{p}_i \Big)$$

$$= \delta \underline{\phi} \cdot \frac{d}{dt} \sum_i \left(\underline{r}_i \times \underline{p}_i \right) = 0$$

so $\delta L = 0 \Rightarrow \dfrac{d}{dt} \underline{L} = 0 \; , \quad \underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

$\Rightarrow$ rotational invariance $\Rightarrow$ angular momentum conserved.

Note: Angular momentum depends on choice of origin, except when system at rest, as a whole

i.e. $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

$\underline{r} \to \underline{r}' + \underline{a}$

$$\underline{L} = \sum_i \underline{r}_i' \times \underline{p}_i + \sum_i \underline{a} \times \underline{p}_i$$

$$= \underline{L}' + \underline{a} \times \underline{P}$$

N.B.: Do problem 3, section 9 of L&L. Approach intuitively $\to$ see below.

Observe:

time homogeneity $\leftrightarrow$ $t \to t_0 + t'$, $\partial_t L = 0$ $\leftrightarrow$ energy conservation

spatial homogeneity $\leftrightarrow$ $\underline{x} \to \underline{x}' + \underline{c}$ $\leftrightarrow$ linear momentum conservation

$$\partial_{\underline{x}} L = 0$$

rotational isotropy $\Rightarrow$ $\underline{\phi} = \underline{\phi}' + \delta\phi$ $\Rightarrow$ angular momentum conservation

$\partial L / \partial \phi = 0$

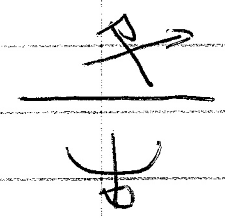
$\Rightarrow$ suggestive of connection between:

symmetry $\rightarrow$ ignorable or 'cyclic' coordinate $\left(\partial L / \partial q = 0\right)$ $\rightarrow$ conservation property/law

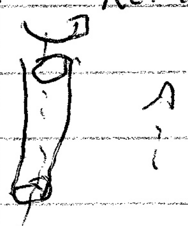
$\Rightarrow$ Noether's Theorem!

Ex. which components $\underline{p}$, $\underline{L}$ conserved in following fields?

a.) inf. homog. plane: 2 components $\underline{p}$ // plane; $\underline{L}_{\perp}$ perp to plane.



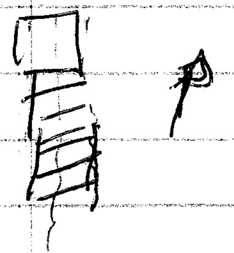
b.) inf. homog. cylinder



$\underline{p}$ component along axis

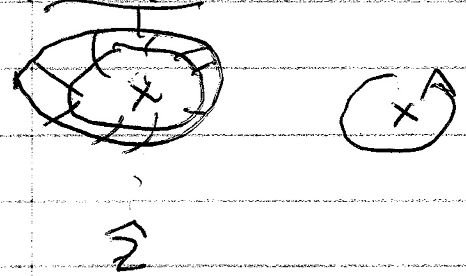
$\underline{L}$ component along axis.

c.) inf. homogeneous rectangular prism



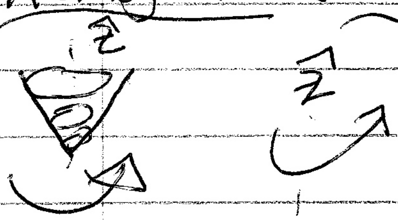
$\rightarrow p_z$, ∥ prism

d.) Torus L_z, only.

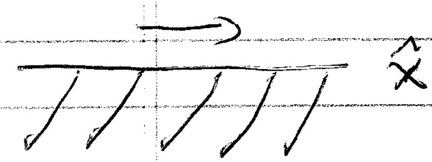


e.) Two points L_x, only

f.) homogeneous cone L_z only

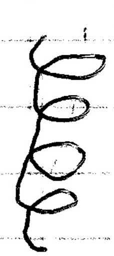


g.) infinite half plane slab



p_x only

h.) cnf. homogeneous cylindrical helix.



$h \equiv$ pitch (vert. dist. in / rotation)

Invariance:

$\delta\phi$ rotation $+ \frac{\delta\phi}{2\pi} h$ vert. translation

i.e. $$\delta L = \frac{\partial L}{\partial z}\delta z + \frac{\partial L}{\partial \phi}\delta\phi$$

$$\delta z = \frac{h}{2\pi}\delta\phi\,, \qquad \partial L/\partial z = \frac{d}{dt}P_z$$

$$\Rightarrow \quad \delta L = \frac{h}{2\pi}\dot{P}_z\,\delta\phi + \dot{L}_z\,\delta\phi$$

$$= \delta\phi\left(\frac{h}{2\pi}\dot{P}_z + \dot{L}_z\right)$$

$$\delta L = 0 \quad \Rightarrow \quad \frac{h}{2\pi}P_z + L_z = \text{const.}$$

Now ..... can also consider scale symmetry $\Rightarrow$ Virial Theorems.

N.B. Virial Thms are simple, interesting and useful $\rightarrow$ esp in astrophysics.

Re: scale symmetry $\Rightarrow$

if $U(\alpha \underline{r}_1, \alpha \underline{r}_2, \dots \alpha \underline{r}_n) = \alpha^k U(\underline{r}_1, \underline{r}_2, \dots \underline{r}_n)$

i.e. re-scale variables

$\Rightarrow$ U is homogeneous. (related to scale invariance)

Many relevant U are homogeneous, i.e.

harmonic oscillator: $k = 2$

Coulomb/Gravity: $k = -1$

etc. $\Rightarrow$ homogeneous U $\Rightarrow$ power law structure.

Now, recall:

EOM from $\delta \int dt\, L = 0$, so if

$L \to \underset{\substack{\downarrow \\ \text{const. factor}}}{\sigma} L$, EOM unchanged.

$\Rightarrow$ Multiplying Lagrangian by constant factor leaves physics unchanged.

→ So, for homogeneous U, can generate class of rescalings which multiply Lagrangian by const. factor

⇒ same EOM.

→ such rescalings define basic class of relations between quantities.

→ useful for basic story/characteristics w/o detailed crank.

Now: $S = \int dt \left(\frac{1}{2} m \dot{\underline{r}}^2 - U(\underline{r}) \right)$

$r \to \alpha r'$

$t \to \beta t'$

$S = \int dt \left(\frac{1}{2} m \frac{\alpha^2}{\beta^2} \dot{r}'^2 - \alpha^k U(r') \right)$

so if $\alpha^2/\beta^2 \sim \alpha^k$ ⇒ L multiplied by factor and δS invariant.

$\Rightarrow$ $\boxed{\begin{array}{l}\beta \sim \alpha^{1-k/2} \text{ defines} \\ \text{time - space rescaling} \\ \text{leaving EOM unchanged.}\end{array}}$

i.e. $\boxed{t'/t \sim (\ell'/\ell)^{1-k/2}}$

$\Rightarrow$ works only for homog. fctn.

Equivalently:

$$\left\{\begin{array}{l} v'/v \sim \alpha^{k/2} \sim (\ell'/\ell)^{k/2} \\ \\ E'/E \sim (\ell'/\ell)^{k} \\ \\ L'/L \sim (\ell'/\ell)^{1+k/2} \end{array}\right.$$

$\rightarrow$ Eg. (a) $U \sim z$ gravity $k \sim 1$

$\Rightarrow$ $t'/t \sim (\ell'/\ell)^{1/2}$

fall time ~ square of amplitude.

(b) $U \sim r^2$ $\quad k = 2$ (h.o.)

$t'/t \sim l^0 \Rightarrow$ period indep. of amplitude.

(c) $U \sim r^{-1}$ $\quad k = -1$

$t'/t \sim (l'/l)^{3/2}$

Period $\sim$ (radius)$^{3/2}$ $\Rightarrow$ Kepler's 3rd Law.

Homogeneous $U \leftrightarrow$ Virial Thms!

What is a Virial Thm?

- consider a system of particles

i

- $$\frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= 2\underset{\substack{\downarrow \\ KE}}{T} - \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i$$

Now consider $\left\langle \left(\sum_i \underline{p}_i \cdot \underline{x}_i \right)^{\bullet} \right\rangle$ → time avg.

$$\langle A \rangle = \frac{1}{\tau} \int_0^{\tau} A \quad \text{as } \tau \to \infty$$

So if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded in time

$$\left\langle \frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right\rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} dt \left(\frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right)$$

$$\to 0$$

so

$$2\langle T \rangle = \left\langle \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Now, if $U = U(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n)$
and U homogeneous

i.e. $U(\alpha \underline{x}_1, \ldots \alpha \underline{x}_n) = \alpha^k U(\underline{x}_1, \ldots \underline{x}_n)$

then $\boxed{2\langle T \rangle = k \langle U \rangle.}$

and also, of course, have:

$$T + U = E = \langle T \rangle + \langle U \rangle$$

so

$$2\langle T \rangle = k\langle U \rangle$$

$$E = \langle T \rangle + \langle U \rangle$$

$$\Rightarrow \boxed{\begin{aligned} \langle U \rangle &= \frac{2}{k+2} E \\ \langle T \rangle &= kE/k+2 \end{aligned}}$$

check:

$\rightarrow$ $k = 2$ $\qquad$ $\langle U \rangle = E/2$, $\langle T \rangle = E/2$

equipartition in H. O.

$\rightarrow$ $k = -1$

$$\langle U \rangle = E$$
$$\langle T \rangle = -E$$

$\leadsto \langle T \rangle = -E$ $\Rightarrow$ total energy negative for gravitationally bound cluster

i.e. must have boundstate for time avg. to converge.

Who cares / why care?

- virial thms relate energies to potential structure
- can relate measured K.E. (Doppler spectroscopy) to energies
- virial is single # characterizing a cluster

$f(\underline{x}, \underline{v}, t) \rightarrow V(\underline{x}, t) \rightarrow \langle T \rangle, \langle U \rangle, E$

Boltzmann $\rightarrow$ fluid $\rightarrow$ virial

$\downarrow$ velocity moment

$\downarrow$ $\sum_i \rightarrow$ integral in space.